Lecture 39

Measure and Integration

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$\{f_n\}_{n \geq 1}$ on $(X, \leq)$

f_n converges to f <u>pointwise</u>

$\Leftrightarrow$ $f_n(x) \longrightarrow f(x) \ \forall \ x \in X$

$f_n \longrightarrow f$ <u>uniformly</u>.

<u>Pointwise</u> $\forall \ x \in X, \ \forall \ \varepsilon > 0,$

$\exists \ n_0(x, \varepsilon)$ such that

$|f_n(x) - f(x)| < \varepsilon \ \forall \ n \geq n_0$

$f_n \longrightarrow f$ uniformly

$\forall x, \forall \varepsilon > 0, \exists n_0(\varepsilon)$
such that

$$f_n(x) \longrightarrow f(x) \text{ i.e.}$$

$$|f_n(x) - f(x)| < \varepsilon \quad \forall \underline{n \geq n_0}$$

Uniform convergence
$\Longrightarrow$ Pointwise convergence.

$f_n \longrightarrow f$ a.e. if $\quad (X, S, \mu)$

$N := \{ x \in X \mid f_n(x) \not\rightarrow f(x) \}$

$N \in S, \ \mu(N) = 0.$

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pointwise convergence

$\Longrightarrow$ convergence a.e.

$\not\Longleftarrow$

$f_n \longrightarrow f$ a.e.

$$N = \{ x \in X \mid f_n(x) \not\longrightarrow f(x) \}$$

Then $\mu(N) = 0$.

~~Let~~ On N^c, $f_n(x) \longrightarrow f(x)\ \forall x \in N$

Let $E_m(S) = \{ x \in X \mid |f_n(x) - f(x)| <$
$\forall n \geq m \}$

Note $E_m(S) \in \mathcal{S}$, $E_m(S) \uparrow$

~~The sequence~~

$$N^c \subseteq \bigcup_{m=1}^{\infty} E_m(\delta)$$

Thus $\{E \cap E_m^c(\delta)\}_{m \geq 1}$ is a decreasing sequence and $\mu(E) < +\infty \Longrightarrow$

$$\lim_{m \to \infty} \mu(E \cap E_m^c(\delta)) = 0$$

Given $\varepsilon > 0$, $\exists\ m_0$ such that

$$\mu(E \cap E_{m_0}^c) < \varepsilon$$

On $E \cap E_{m_0}$,

$$|f_n(x) - f(x)| < \delta \quad \forall\ n \geq m_0$$

$f_n \longrightarrow f$ a.e, $\mu(E) < +\infty$.

$\forall \ \delta = \frac{1}{m}$, $\exists$ a set $E_m \subseteq E$, n_m

Such that

$$\mu(E \setminus E_m) < \varepsilon/2^m$$

and on E_m,

$$|f_n(x) - f(x)| < \frac{1}{m} \ \forall\, n \geq n_m$$

Define $\quad E_\varepsilon = \bigcap_{m\geq 1}^{\infty} E_m .$

$$\mu(E \setminus E_\varepsilon) = \mu\left(\bigcup_{m=1}^{\infty} (E \setminus E_m)\right)$$

$$\leq \sum_{m=1}^{\infty} \mu(E \setminus E_m)$$

$$\leq \sum_{m=1}^{\infty} \varepsilon/2^m \leq \varepsilon$$

$$\mu(E \setminus E_\varepsilon) < \varepsilon.$$

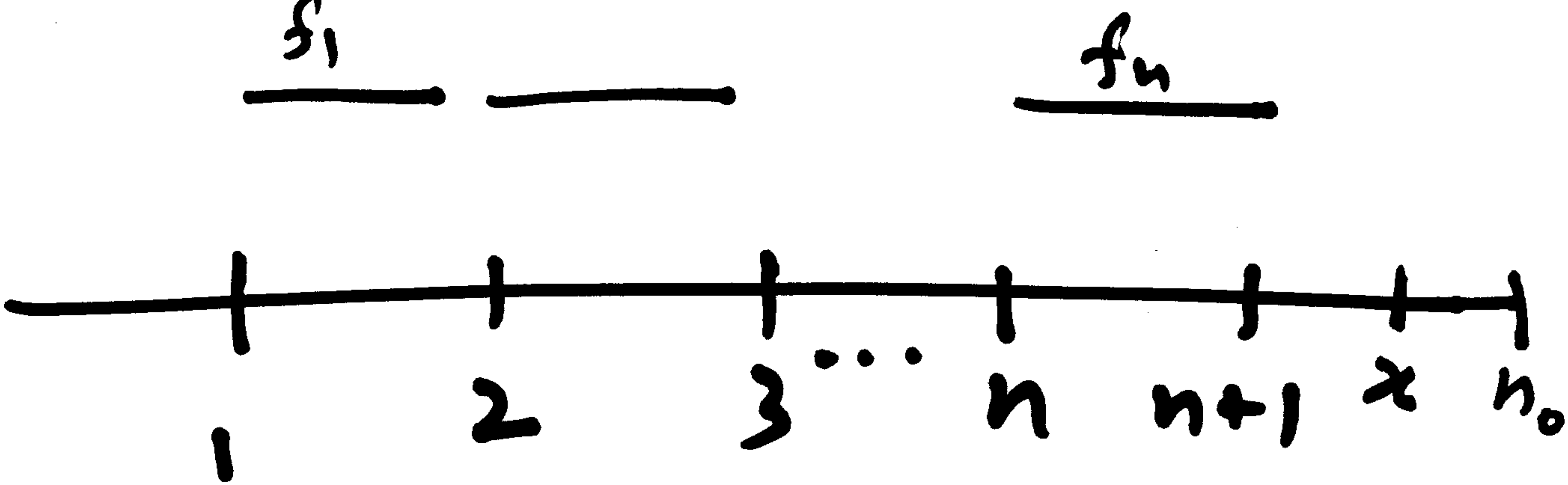
Also, $x \in E_\varepsilon = \bigcap_{m=1}^{\infty} E_m$

$$\Rightarrow x \in E_m \ \forall m$$

$$|f_n(x) - f(x)| < \frac{1}{m} \quad \forall\, n \geq n_m$$

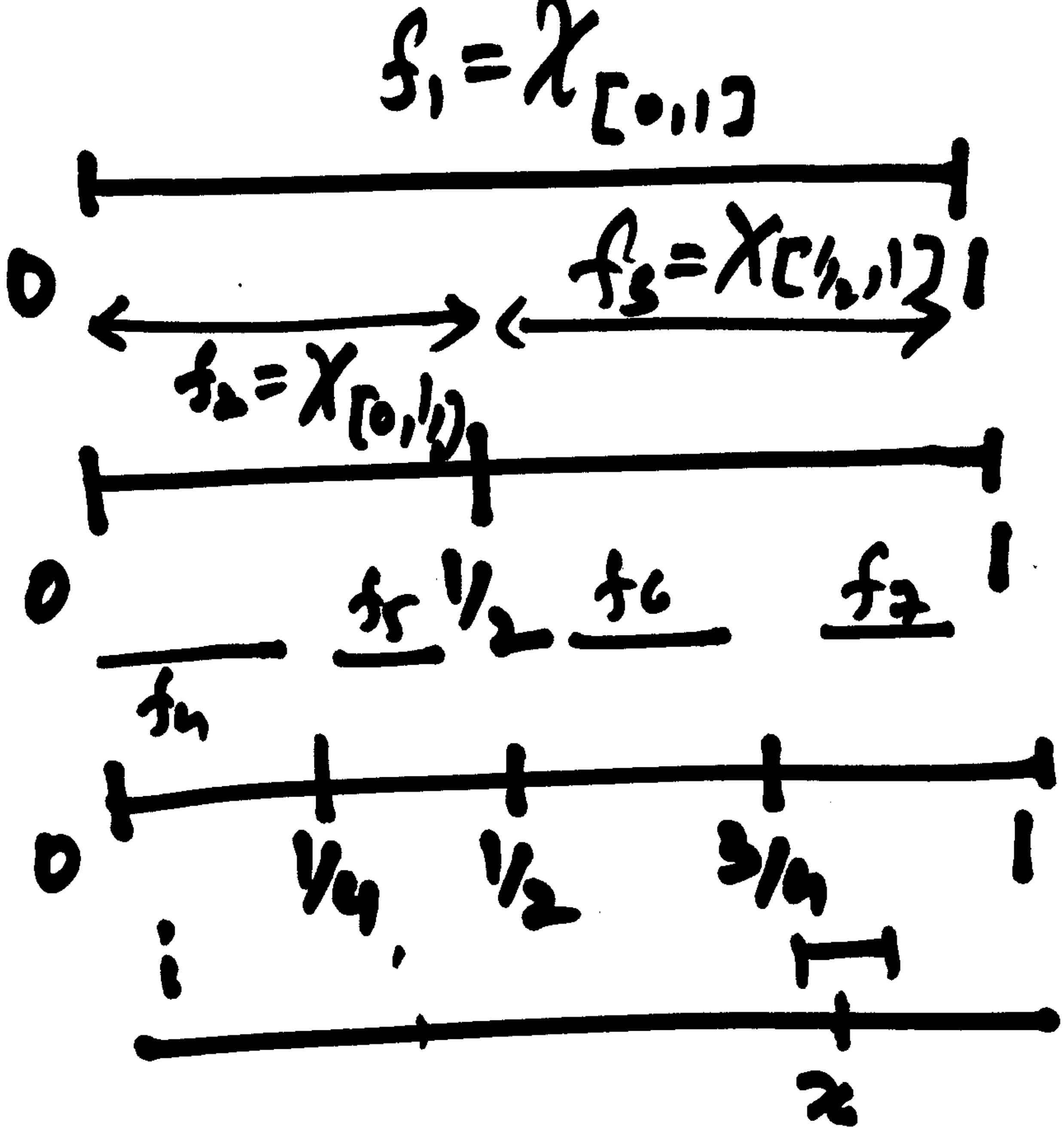
$$\Rightarrow \quad f_n \xrightarrow{u} f \text{ on } E_\varepsilon$$

$$f_n = \chi_{[n, n+1]}$$



$$f_n(x) \longrightarrow f(x) \quad \forall x$$

$$f_{n_0}(x) = 0 = f(x).$$

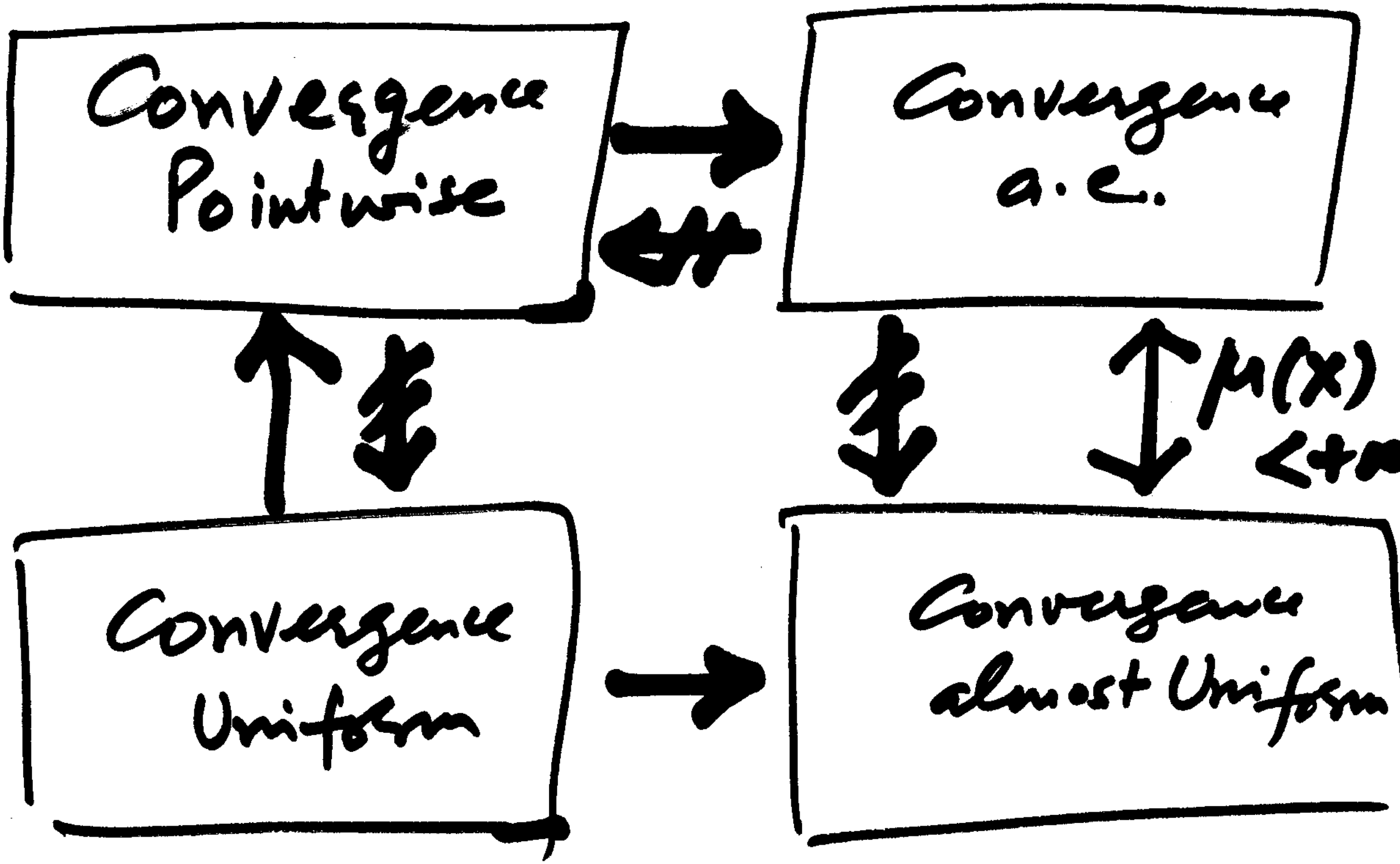


$f_n = \chi_{I_n^k}$

$\lambda(I_n^k) = \frac{1}{2^n}$

$f_n \xrightarrow{m} f \equiv 0$

$f_n \not\to f$ pointwise


Convergence Pointwise
Convergence a.e.
Convergence Uniform
Convergence almost Uniform
$\mu(X) < +\infty$